Review on Several Nonlinear Mappings (Nonlinear Analysis and Convex Analysis)

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Abstract

The purpose of this paper is systematically to survey several nonlinear mappings and to classify the implications concerning to them. Also, we suggest several examples of nonlinear mappings which are comparable each other and we finally raise some open questions.

Keywords: non-Lipschitzian mapping, total asymptotically nonexpansive mapping, generalized projection, relatively total asymptotically nonexpansive, generalized total asymptotically nonexpansive.

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1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. Also, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and the set of real numbers, respectively. Let $T : C \to C$ be a mapping. We denote by $F(T)$ the set of all fixed points of T, namely,

$$F(T) = \{x \in C : Tx = x\}.$$

Recall that the mapping T is said to be *Lipschitzian* if

$$\|Tx - Ty\| \le L\|x - y\|, \quad x, y \in C,$$

where $L := L_T$ denotes the *Lipschitz constant* of T. Obviously, it is equivalent to the following property: for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad x, y \in C. \tag{1.1}$$

For a Lipschitzian mapping T, we say:

- T is *uniformly k-Lipschiztain* if $k_n = k$ for all $n \in \mathbb{N}$;
- T is *nonexpansive* if $k_n = 1$ for all $n \in \mathbb{N}$;
- T is *asymptotically nonexpansive* [3] if $\lim_{n\to\infty} k_n = 1$.

The first non-Lipschitzian mapping was introduce by Kirk [10]; we say that T is a mapping of *asymptotically nonexpansive type* if

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0 \tag{1.2}$$

for every $x \in C$, and T^N is continuous for some $N \geq 1$. In 1993, Bruck et al [2] introduced the stronger definition than (1.2), namely, T is said to be *asymptotically nonexpansive in the intermediate sense* [2] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \leq 0. \tag{1.3}$$

Note that if we define

$$c_n := \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0, \tag{1.4}$$

where $a \vee b := \max\{a, b\}$, then (1.3) ensures that $c_n \to 0$ and

$$\|T^n x - T^n y\| \leq \|x-y\| + c_n \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$. Obviously, (1.5) implies (1.3) in case $c_n \to 0$. Therefore, we summarize:

Proposition 1.1. *(1.3) holds $\Leftrightarrow$ (1.5) holds for some sequence $\{c_n\}$ with $c_n \to 0$.*

Definition 1.2. We say that T is gradually nonexpansive whenever (1.5) holds for some sequence $\{c_n\}$ with $c_n \to 0$.

For the purpose of unifying nonlinear mappings mentioned above, Alber et al [1] introduced a new notion, namely, T is said to be *total asymptotically nonexpansive* [1] if there exist two nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n, \beta_n \to 0$, $\tau \in \Gamma[0,\infty)$ such that

$$\|T^n x - T^n y\| \leq \|x-y\| + \alpha_n \tau(\|x-y\|) + \beta_n, \quad x, y \in C,\ n \geq 1, \tag{1.6}$$

where $\tau \in \Gamma[0,\infty)$ if and only if $\tau : [0,\infty) \to [0,\infty)$ is strictly increasing, continuous on $[0,\infty)$ and $\tau(0) = 0$.

Remark 1.3. In view of Definition 3.1 of [12], we also say that T is generalized asymptotically nonexpansive in case that $\tau(t) = t$ for all $t \geq 0$ in (1.6).

Now it is natural to consider more stronger one than (1.6).

Definition 1.4. T is said to be *square total asymptotically nonexpansive* if (1.6) can be replaced by

$$\|T^n x - T^n y\|^2 \leq \|x-y\|^2 + \widetilde{\alpha}_n \widetilde{\tau}(\|x-y\|^2) + \widetilde{\beta}_n, \tag{1.7}$$

for all $x, y \in C$ and $n \geq m_0$, where $m_0 \in \mathbb{N}$, $\widetilde{\alpha}_n, \widetilde{\beta}_n \to 0$ and $\widetilde{\tau} \in \Gamma[0,\infty)$.

Remark 1.5. Note that the property (1.6) with $\alpha_n = 0$ for all $n \geq 1$ reduces to (1.5) with $\beta_n = c_n$; moreover, if we take $\tau(t) = t$ for all $t \geq 0$ and $\beta_n = 0$ for all $n \geq 1$ in (1.6), it is reduced to (1.1) with $k_n = 1 + \alpha_n$.

The purpose of this paper is systematically to survey several nonlinear mappings and to classify the implications concerning to them. In section 2, we introduce classes of various nonlinear mappings and suggest their implications. In section 3, we give some counter examples for their implications and some open questions are finally added.

2 Implications of classes

For summarizing the connections between the classes of nonlinear mappings considered above, we use the following notations:

$$
\begin{aligned}
\text{(N)} &= \text{the class of nonexpansive mappings}\\
\text{(L)} &= \text{the class of Lipschiztian mappings}\\
\text{(UC)} &= \text{the class of uniformly continuous mappings}\\
\text{(UL)} &= \text{the class of uniformly Lipschiztian mappings}\\
\text{(AN)} &= \text{the class of asymptotically nonexpansive mappings}\\
\text{(GN)} &= \text{the class of gradually nonexpansive mappings}\\
\text{(GAN)} &= \text{the class of generalized asymptotically nonexpansive mappings}\\
\text{(TAN)} &= \text{the class of total asymptotically nonexpansive mappings}\\
\text{(STAN)} &= \text{the class of square total asymptotically nonexpansive mappings}\\
\text{(ANIS)} &= \text{the class of mappings which are asymptotically nonexpansive}\\
&\ \text{in the intermediate sense}\\
\text{(ANT)} &= \text{the class of mappings of asymptotically nonexpansive type}
\end{aligned}
$$

We say that T is AN, GN, GAN, TAN, STAN, ANIS and ANT, in abbreviated forms, whenever T belongs to its corresponding classes (AN), (GN), (GAN), (TAN), (ANIS) and (ANT). Then, there hold the following implications:

Proposition 2.1. (i) (N) $\subset$ (AN) $\subset$ (UL) $\subset$ (L) $\subset$ (UC).

(ii) (GN) $\cap$ (UC) = (ANIS) $\subset$ (ANT).

(iii) (AN) $\cup$ (GN) $\subset$ (GAN) $\subset$ (TAN).

Proposition 2.2. *Let C be bounded. Then:*

(i) (GN) = (GAN) = (TAN) = (STAN).

(ii) (AN) $\subset$ (ANIS).

Proof. (i) Since (TAN) = (STAN) is obvious, we claim: (TAN) $\subset$ (GN). Assume $\delta :=$ diam$(C) < \infty$. If T is TAN, then

$$
\begin{aligned}
\|T^n x - T^n y\| &\leq \|x-y\| + \alpha_n \tau(\|x-y\|) + \beta_n\\
&\leq \|x-y\| + \alpha_n \tau(\delta) + \beta_n\\
&= \|x-y\| + c_n, \quad x, y \in C,\ n \geq 1,
\end{aligned}
$$

where $c_n := \alpha_n \tau(\delta) + \beta_n \delta \to 0$. Hence T is GN. It follows from (iii) of Proposition 2.1 that (GN) = (GAN) = (TAN). Moreover, since (AN) $\cup$ (GN) = (GN), we have (AN) $\subset$ (GN). Then it follows from (i) and (ii) of Proposition 2.1 that (AN) $\subset$ (ANIS). Therefore (ii) is obtained. $\square$

Definition 2.3. Let f_n, $f : C(\subset X) \to C$ be mappings. We say that $\{f_n\}$ converges uniformly to f on C if

$$\|f_n - f\| := \sup_{x\in C} \|f_n(x) - f(x)\| \to 0$$

as $n \to \infty$. Further, we say that $T^n x$ converges uniformly to a point $p \in C$ on C whenever $\{f_n := T^n\}$ converges uniformly to the function f on C, where $f(x) = p$ for all $x \in C$, that is,

$$\sup_{x\in C} \|T^n x - p\| \to 0 \quad \text{as } n \to \infty.$$

Lemma 2.4. *If $T^n x$ converges uniformly to some point $p \in C$ on C, then T is GN.*

Proof. Let $c_n := \sup_{x,y\in C} \|T^n x - T^n y\|$. Then $c_n \to 0$ since

$$0 \le c_n = \sup_{x,y\in C} \|T^n x - T^n y\| \le \sup_{x\in C} \|T^n x - p\| + \sup_{y\in C} \|p - T^n y\| \to 0$$

as $n \to \infty$. From the construction of c_n it easily follows that

$$\|T^n x - T^n y\| \le c_n, \quad x, y \in C,\ n \ge 1,$$

which immediately implies (1.5). Hence T is GN. □

Remark 2.5. However, the converse of Lemma 2.4 does not hold in general; see Example 3.9.

3 Counter examples

As counter examples of (i) in Proposition 2.1, it is not hard to see that if $Tx = 2x$ for $x \in C = \mathbb{R}$, then $T \in$ (L)\ (UL). Furthermore, if $Tx = \sqrt{x}$ for $x \in C = [0, \infty)$, then $T \in$ (UC)\ (L). For an example of $T \in$ (UL)\ (AN), see Example 3.7. Also, the following example of $T \in$ (AN) \ (N) is originally due to [3] in ℓ^2 spaces.

Example 3.1. ([7]; see Example 3.13). *Let B denote the unit ball in the space $X = \ell^p$, where $1 < p < \infty$. Obviously, X is uniformly convex and uniformly smooth. Let $T : B \to B$ be defined by*

$$Tx = (0, x_1^2, \lambda_1 x_2, \lambda_2 x_3, \dots)$$

for all $x = (x_1, x_2, x_3, \dots) \in B$, where $0 < \lambda_n < 1$ for all $n \ge 1$ and $\prod_{n=1}^{\infty} \lambda_n = \frac{1}{2}$. Then:

(i) *T is Lipschitzian, i.e., $\|Tx - Ty\| \le 2\|x - y\|, \quad x, y \in C;$*

(ii) *T is AN, i.e., $\|T^{n+1}x - T^{n+1}y\| \le 2\prod_{i=1}^{n-1} \lambda_i \|x - y\|, \quad x, y \in C,\ n \in \mathbb{N};$*

(iii) *T is not nonexpansive.*

Proof. Noticing that, for $x = (x_1, x_2, \dots) \in B$,

$$T^n x = \Big(\overbrace{0, \dots, 0}^{n}, \prod_{i=1}^{n-1} \lambda_i x_1^2, \prod_{i=1}^{n} \lambda_i x_2, \prod_{i=2}^{n+1} \lambda_i x_3, \dots\Big).$$

Thus we have $\|T^n x - T^n y\| \le 2\prod_{i=1}^{n-1} \lambda_i \|x - y\|$ for all $n \ge 2$. Obviously, since $2\prod_{i=1}^{n-1} \lambda_i \downarrow 1$, T is AN. On the other hand, since $\|Tx - Ty\| = \frac{3}{4} > \frac{1}{2} = \|x - y\|$ for $x = (1, 0, 0, \dots)$ and $y = (1/2, 0, 0, \dots)$, T is not nonexpansive. □

Remark 3.2. Consider either $\lambda_n := 1 - \frac{1}{(n+1)^2}$ or $\lambda_n := \exp\left(\frac{1}{2n} - \frac{1}{2n-1}\right)$ to get a sequence satisfying that $0 < \lambda_n < 1$ and $\prod_{n=1}^{\infty} \lambda_n = \frac{1}{2}$. Indeed, for the second case, since $0 < \exp(-x) < 1$ for all $x > 0$, we must find a sequence $\{\alpha_n\}$ such that

$$\prod_{n=1}^{\infty} \lambda_n = \prod_{n=1}^{\infty} \exp(-\alpha_n) = \exp(-\sum_{n=1}^{\infty} \alpha_n) = \frac{1}{2}.$$

This is equivalent to

$$\sum_{n=1}^{\infty} \alpha_n = \ln 2 = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

because

$$\begin{aligned} s_n &:= \sum_{k=1}^{n}\Big(\frac{1}{2k-1}-\frac{1}{2k}\Big) = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n} \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+k/n} \to \int_0^1 \frac{1}{1+x}dx = \ln 2. \end{aligned}$$

Furthermore, since $1-x \le \exp(-x)$ for all $x \in (0,1)$, we observe

$$\sum_{n=1}^{\infty}\alpha_n = \infty \;\Rightarrow\; \prod_{n=1}^{\infty}(1-\alpha_n) = 0$$

for all $\alpha_n \in (0,1)$ since

$$0 \le \prod_{n=1}^{\infty}(1-\alpha_n) \le \prod_{n=1}^{\infty}\exp(-\alpha_n) = \exp(-\sum_{n=1}^{\infty}\alpha_n) = 0.$$

As a direct consequence of Lemma 2.4, we introduce three non-Lipschitzian mappings $T \in$ (ANIS) $\setminus$ (AN).

Example 3.3. ([5]). *Let $C := \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$ and $0 < |k| < 1$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx := \begin{cases} kx\sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Then $F(T) = \{0\}$ and $T^n x$ converges uniformly to 0 on C (hence T is GN by Lemma 2.4). Since T is clearly uniformly continuous, it follows from (ii) of Proposition 2.1 that T is ANIS. However, T is not Lipschitzian; see Example 4.3 of [5] for the proof. Therefore $T \in$ (ANIS) $\setminus$ (AN).

Example 3.4. ([9]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} \alpha, & x \in [0,\alpha]; \\ \frac{\alpha}{\sqrt{1-\alpha}}\sqrt{1-x}, & x \in [\alpha, 1], \end{cases}$$

where $\alpha \in (0,1)$. Then $F(T) = \{\alpha\}$ and $T^n x = \alpha$ for all $x \in C$, $n \ge 2$; hence T is ANIS as in Example 3.3. However, T is not Lipschitzian; see Example 3.9 of [9] for the proof. Therefore $T \in$ (ANIS) $\setminus$ (AN).

Example 3.5. ([4]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} (\sqrt{2}-1)\sqrt{\frac{1}{2}-x}+\frac{1}{\sqrt{2}}, & \text{if } 0 \le x \le 1/2; \\ \sqrt{x}, & \text{if } 1/2 \le x \le 1. \end{cases}$$

Then $F(T) = \{1\}$ and $T^n x$ converges uniformly to 1 on C; hence T is ANIS as in Example 3.3. However, T is not Lipschitzian; see Example 1.2 of [4] for more details. Therefore $T \in$ (ANIS) $\setminus$ (AN).

A mapping satisfying the property (1.3) do not always guarantee its *non-Lipschitz*. The following two examples are *uniformly Lipschitzian* ANIS mappings.

Example 3.6. ([8]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} kx, & \text{if } 0 \le x \le 1/2; \\ \frac{k}{2k-1}(k-x), & \text{if } 1/2 \le x \le k; \\ 0, & \text{if } k \le x \le 1, \end{cases}$$

where $1/2 < k < 1$. Then $F(T) = \{0\}$ and $T^n x$ converges uniformly to 0 on C. Obviously, T is uniformly continuous. It follows from Lemma 2.4 and (ii) of Proposition 2.1 that T is ANIS. Furthermore, T is uniformly Lipschitzian. Indeed, if $0 \le x \le 1/2$ and $1/2 \le y \le k$, then $T^n x = k^n x$ and $T^n y = \frac{k^n}{2k-1}(k-y)$. Therefore, we see that

$$\begin{aligned} |T^n x - T^n y| &= \left| k^n x - \frac{k^n}{2} + \frac{k^n}{2} - \frac{k^n}{2k-1}(k-y) \right| \\ &= \left| k^n\left(x - \frac{1}{2}\right) + \frac{k^n}{2k-1}\left[\left(k - \frac{1}{2}\right) - (k-y)\right] \right| \\ &\le k^n \left| x - \frac{1}{2} \right| + \frac{k^n}{2k-1}\left| y - \frac{1}{2} \right| \\ &\le \frac{k^n}{2k-1}|x-y| \le \frac{k}{2k-1}|x-y|. \end{aligned}$$

The remaining cases are obvious. Hence T is uniformly $\frac{k}{2k-1}$-Lipschitzian.

The following example of $T \in (\mathrm{UL}) \cap (\mathrm{ANIS}) \setminus (\mathrm{AN})$ is originally due to Example 1.3 (with $k = 4$) of [4].

Example 3.7. ([4]). *For any $k > 0$, let $\{a_n\}$ be a sequence of positive numbers such that $a_n \downarrow 0$ and $\prod_{n=1}^{\infty}(1 + a_n) = k$. Set*

$$b_n := \frac{1}{2^{n+1}(1+a_n)}, \quad n \ge 1.$$

Let $T : C \to C$ be defined by

$$Tx = \begin{cases} (1+a_1)x + 1/2, & \text{if } x \in [0, b_1]; \\ 1/2 + 1/4, & \text{if } x \in [b_1, 1/2] \end{cases}$$

and

$$Tx = \begin{cases} (1+a_n)\Big(x - \sum\limits_{i=1}^{n-1} \frac{1}{2^i}\Big) + \sum\limits_{i=1}^{n} \frac{1}{2^i}, & \text{if } x \in \Big[\sum\limits_{i=1}^{n-1} \frac{1}{2^i}, \sum\limits_{i=1}^{n-1} \frac{1}{2^i} + b_n\Big]; \\ \sum\limits_{i=1}^{n+1} \frac{1}{2^i}, & \text{if } x \in \Big[\sum\limits_{i=1}^{n-1} \frac{1}{2^i} + b_n, \sum\limits_{i=1}^{n} \frac{1}{2^i}\Big], \quad n \ge 2 \end{cases}$$

and $T1 = 1$. Then $F(T) = \{1\}$ and $T^n x$ converges uniformly to 1 on C. Since T is continuous on C, T is also uniformly continuous on C. It follows from Lemma 2.4 and (ii) of Proposition 2.1 that T is ANIS. Furthermore, T is uniformly k-Lipschitzian. Note that if $k > 1$, we conclude : $T \in (\mathrm{UL}) \cap (\mathrm{ANIS}) \setminus (\mathrm{AN})$.

Proof. It suffices to show that T is uniformly k-Lipschitzian. Indeed, (i) if we take

$$x := \prod_{i=1}^{n} b_i = \frac{1}{\prod_{i=1}^{n} 2^{i+1} \cdot \prod_{i=1}^{n}(1+a_i)}, \quad y := \frac{b_n}{\prod_{i=1}^{n-1}(1+a_i)}$$

for each $n \ge 1$, then, since

$$T^n x = \prod_{i=1}^{n}(1+a_i)x + \sum_{i=1}^{n} \frac{1}{2^i}, \quad T^n y = \sum_{i=1}^{n+1} \frac{1}{2^i}$$

and $\prod_{k=1}^{n}(1+a_k)y = b_n(1+a_n) = \frac{1}{2^{n+1}}$, we obtain that

$$\begin{aligned}
|T^n x - T^n y| &= \Big|\prod_{i=1}^{n}(1+a_i)x + \sum_{i=1}^{n}\frac{1}{2^i} - \sum_{i=1}^{n+1}\frac{1}{2^i}\Big| \\
&= \Big|\prod_{i=1}^{n}(1+a_i)x - \frac{1}{2^{n+1}}\Big| \\
&= \Big|\prod_{i=1}^{n}(1+a_i)x - b_n(1+a_n)\Big| = \prod_{i=1}^{n}(1+a_i)|x-y|.
\end{aligned}$$

(ii) if we replace y with a point $p \in [b_1, 1/2]$ in the case (i), then $T^n p = \sum_{i=1}^{n+1}\frac{1}{2^i}$. Now use $1/2^2 = (1+a_1)b_1$ and $\prod_{i=1}^{n}(1+a_i)x = 1/(\prod_{i=1}^{n}2^{i+1})$ to derive

$$\begin{aligned}
|T^n x - T^n p| &= |T^n x - T^n b_1| \\
&= \Big|\prod_{i=1}^{n}(1+a_i)x + \sum_{i=1}^{n}\frac{1}{2^i} - \sum_{i=1}^{n+1}\frac{1}{2^i}\Big| \\
&= \Big|\prod_{i=1}^{n}(1+a_i)x - \frac{1}{2^{n+1}}\Big| = \frac{1}{2^{n+1}} - \frac{1}{\prod_{i=1}^{n}2^{i+1}} \\
&\le \frac{\prod_{i=2}^{n}(1+a_i)}{2^2} - \frac{1}{\prod_{i=1}^{n}2^{i+1}} \\
&= \prod_{i=1}^{n}(1+a_i)b_1 - \prod_{i=1}^{n}(1+a_i)x = \prod_{i=1}^{n}(1+a_i)|x-b_1| \\
&\le \prod_{i=1}^{n}(1+a_i)|x-p|.
\end{aligned}$$

The remaining cases are obvious. We conclude that

$$|T^n x - T^n y| \le \prod_{i=1}^{n}(1+a_i)|x-y|, \qquad x, y \in C,\ n \ge 1,$$

which ensures that T is uniformly k-Lipschiztian since $\prod_{n=1}^{\infty}(1+a_n) = k$. □

Remark 3.8. (i) Since $1 + x \le e^x$ for all $x \in \mathbb{R}$, we easily find a sequence $\{a_n\}$ of positive numbers such that $a_n \downarrow 0$ and $\prod_{n=1}^{\infty}(1+a_n) = k (> 1)$. Indeed, since the sequence $\{s_n\}$, $s_n := \prod_{k=1}^{n}(1+a_k)$, of its nth partial sums is strictly increasing and

$$\prod_{n=1}^{\infty}(1+a_n) \le \prod_{n=1}^{\infty} e^{a_n} = e^{\sum_{n=1}^{\infty} a_n} = e^{\ln k} = k,$$

it suffices to find a (convergent) geometric series replaced with $a_n := r^n$, $0 \le r < 1$ such that

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} r^n = \frac{r}{1-r} = \ln k \\
\Leftrightarrow \quad & r = \frac{\ln k}{1+\ln k}.
\end{aligned}$$

(ii) Note that if we take $a_n \equiv 0$ in Example 3.7, then T is clearly nonexpansive.

As a slight modification of Example 3.7, we shall give an example of a uniformly Lipschitzian ANIS mapping defined a (unbounded) closed convex subset C on which is not converges uniformly.

Example 3.9. *Consider $C := [0, \infty) \subset \mathbb{R}$. Let T be defined on $[0,1]$ as in Example 3.7 and define $Tx = x$ on $[1, \infty)$. Since $T^n x$ converges uniformly to 1 on $[0,1]$, setting $c_n :=$ $\sup\{|T^n x - T^n y| : x, y \in [0,1]\} \to 0$, then it is not hard to see that*

$$|T^n x - T^n y| \leq |x - y| + c_n, \quad x, y \in C,\ n \geq 1.$$

Therefore T is GN. In view of Example 3.7, T is uniformly k-Lipschitzian. Therefore, $T : C \to C$ is ANIS. However, note that $T^n x$ does not converge uniformly to a point $p \in C$ on C.

Proof. We claim : $|T^n x - T^n y| \leq |x - y| + c_n$ for all $x, y \in C$ and $n \geq 1$. Consider the case of $x \in [0,1]$ and $y \in [1, \infty)$. Then

$$\begin{aligned} |T^n x - T^n y| &\leq |T^n x - T^n 1| + |T^n 1 - T^n y| \\ &\leq c_n + |1 - y| \leq |x - y| + c_n \end{aligned}$$

for all $n \geq 1$. The remaining cases are obvious. □

The following example of $T \in$ (UL) $\setminus$ (ANT) is also interesting.

Example 3.10. ([6]). *Let $X := \mathbb{R}$, $C := [-\frac{1}{k}, 1]$, where $1 < k < 2$. Define a mapping $T : C \to C$ by*

$$Tx := \begin{cases} -kx, & \text{if } -\frac{1}{k} \leq x \leq 0; \\ -\frac{1}{k}x, & \text{if } 0 \leq x \leq 1. \end{cases}$$

Then:

(i) *$T^2 x = x$ for all $x \in C$ (hence, $T^{2n-1} = T$ for all $n \geq 1$);*

(ii) *T is uniformly k-lipschitzian;*

(iii) *T does not satisfy (1.2); hence it is not ANT.*

Indeed, it suffice to show: T is not ANT. To this end, for each $x \in C$,

$$\begin{aligned} &\limsup_{n \to \infty} \sup_{y \in C} \{|T^n x - T^n y| - |x - y|\} \\ \geq\ & \sup\{|Ty| - |y| : y \in [-1/k, 1]\} \\ =\ & \sup\{(k-1)|y| : -1/k \leq y \leq 0\} \\ =\ & (k-1)\frac{1}{k} = 1 - \frac{1}{k} > 0. \end{aligned}$$

Now we introduce a discontinuous mapping $T \in$ (GN); see Example 1.6 of [11].

Example 3.11. ([11]). *Let $X := \mathbb{R}$, $C := [0,1]$ and $0 < k < 1$. Define a mapping $T : C \to C$ by*

$$Tx := \begin{cases} kx, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ 0, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then T is not continuous at $\frac{1}{2}$ but $T^n x$ converges uniformly to 0 on C. By Lemma 2.4 we readily see that T is GN.

Finally we raise the following questions.

Question 3.12. *Find examples of the following mapping T:*

(i) $T \in$ (ANT) $\setminus$ (ANIS).

(ii) $T \in$ (ANT) $\setminus$ (ANIS).

(iii) $T \in$ (GAN) $\setminus$ (GN).

Question 3.13. *Find the corresponding analogue examples in infinite dimensional spaces.*

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